On Surplus-Sharing in Partnerships*

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September 2, 2014

*We would like to thank Jordi Brandts, Benjamin Brooks, İsa Hafalır, Kevin Hasker, Colin Raymond and seminar participants at Bilkent University, TED University, The 2014 Conference of the Society for Social Choice and Welfare, and The 2014 Murat Sertel Workshop for comments and suggestions. This research was funded in part by the Scientific and Technological Research Council of Turkey (TUBITAK) under grant 114K060. All errors are ours.

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Abstract

We analyze partnerships in which measures of the partners' contributions are available and the success of the partnership is determined stochastically. We consider a family of partnership agreements commonly used in real life. They allocate a fraction ρ of the surplus equally and the remaining $(1-\rho)$ proportional to contributions; and they allow ρ to depend on whether the surplus is positive or negative. We analyze the implications of a partnership agreement on (i) whether the *partnership forms* in the first place, and if it does, (ii) the partners' *contribution choices* as well as (iii) their resulting *welfare*. We then inquire which partnership agreements are *productively efficient* (*i.e.* maximizes the partners' total contributions) and which are *socially efficient*, (*i.e.* maximizes the partners' social welfare as formulated by the two seminal measures of *egalitarianism* and *utilitarianism*).

Keywords: partnership agreements, surplus-sharing, noncooperative contribution choice, proportional surplus shares, equal surplus shares, productive efficiency, egalitarian social welfare, utilitarian social welfare.

JEL classification codes: C72, D86, L22, J33

1 Introduction

Imagine a group of lawyers forming a partnership or a group of investors partnering up to undertake a financial endeavor. As a first step, the partners need to agree on (i) how to allocate positive surplus in case of profits and (ii) how to allocate negative surplus in case of losses. This is a very important choice for the partnership since it in turn affects the partners' contributions as well as their resulting welfare from the partnership. In this paper, we focus on the implications of this choice. More specifically, we analyze the advantages / disadvantages of some *partnership agreements* that are commonly used in real life *professional service partnerships* (such as in law, accounting, medicine, or real estate) as well as *investment partnerships*.

Farrell and Scotchmer (1988) and Lang and Gordon (1995) describe three basic systems law partnerships use to allocate surplus. In the first one, called the *lock-step system*, all partners of the same seniority receive the same surplus share. The lock-step system is used by most 2 or 3 partner law firms, which approximately constitute 2/3 of all law firms in the US, though less than half the lawyers (also see Curran, 1985; Flood, 1985). In the second system, called the *subjective performance-related system*, the firm's founders evaluate each partner's contribution. The partners' surplus shares are then determined in proportion to these evaluations. The third basic system, called an *objective performance-related system*, is only different from the second in the sense that an explicit formula (using variables such as the number of hours billed, cases won, or business brought in) is used to determine each partner's contribution.

The above case of law partnerships demonstrates the two most common surplus sharing methods in real life: *equal* (or in general, fixed) shares versus shares *proportional* to contributions (also called the piece-rate). Gaynor and Pauly (1990) mention that it is also common in professional service partnerships to mix between these two methods by allocating a fixed fraction of the surplus equally and allocating the rest proportional to contributions as a bonus. A partnership agreement can additionally fix different fractions in cases of positive and negative surplus. [1] The following is an example of such a partnership agreement:

Partners Johnson and Smith agree that (i) if their partnership makes a pos-

[1] Legal regulations on partnerships recognize the usage of different surplus sharing rules (as in fixing different fractions) in cases of positive and negative surplus (e.g. see McMasters', 2013).

itive surplus, 60% of this positive surplus will be allocated equally while the remaining 40% will be allocated in proportion to each partner's contribution and (*ii*) if their partnership makes a negative surplus, all of this negative surplus will be allocated equally.

The frame of our study is as follows. First, we take a partnership agreement as a pair of surplus allocation rules, one used for positive, and the other for negative surplus. And we focus on the class of rules discussed in the previous paragraph. Second, we analyze environments where measures of the partners' contributions are available. As already exemplified for law partnerships, such measures are commonly used in professional service partnerships.[2] Similarly, monetary contributions are routinely used to allocate surplus in investment partnerships. Third and last, we assume that the success of the partnership is determined stochastically. That is, whether the partnership makes positive or negative surplus depends on some factors external to it, such as the state of the economy or the performance of the competitors. Several previous studies on partnerships make a similar assumption (e.g. Morrison and Wilhelm (2004), Comino, Nicolo, Tedeschi (2010), Li and Wolfstetter (2010)).[3]

In the confines of this framework, we analyze the implications of a partnership agreement on (*i*) whether the *partnership forms* in the first place, and if it does, (*ii*) the partners' *contribution choices* as well as (*iii*) their resulting *welfare*. Armed with these answers, we then inquire which partnership agreements are *productively efficient* (Gaynor and Pauly, 1990), that is, maximizes the partners' total contributions. We also inquire which partnership agreements are *socially efficient*, that is, maximizes the partners' social welfare as formulated by the two seminal measures of *egalitarianism* and *utilitarianism*. We also analyze the trade-offs between achieving these objectives.

In many countries, legal regulations include a *partnership act*, that is, a statutory agreement that applies to any partnership that does not have a written agreement. This default

[2] For example, number of hours billed in accounting partnerships or number of patients treated in medical partnerships both serve as measures of a partner's contribution.

[3] Our specification serves as a benchmark to a more realistic extension of this model where the probability of success depends on the partners' contributions as well as external factors. The analysis of this extension is left for future research, mainly for lack of a natural way to incorporate contributions into the probability of positive surplus.

agreement typically allocates both positive and negative surplus equally. Also, if the partners have only specified the surplus-sharing rule to be used in case of positive surplus, the legal default is that the same surplus-sharing rule is used in case of negative surplus as well (and vice versa). In the discussion section, we will evaluate these legal practices in reference to our findings.

To this end, we analyze a simple *"partnership game"* in which a group of partners simultaneously choose their contributions. In doing so, each partner tries to maximize his expected utility from the partnership and he is fully informed about the parameters of the game (including the partnership agreement) as well as the others' characteristics. The nature then determines whether the partnership makes a positive or a negative surplus. Depending on the type of surplus realized, the relevant part of the partnership agreement is then used to compensate the partners.

The paper is organized as follows. In *Subsection 1.1*, we discuss the related literature. In *Section 2*, we present the model. The following section contains our findings. In *Subsection 3.1*, we analyze partnership formation. In *Subsection 3.2*, we characterize the equilibrium contributions in a formed partnership. In *Subsection 3.3*, we compare partnership agreements in terms of the total contributions and in *Section 3.4*, we compare them in terms of social welfare. We summarize our findings and conclude in *Section 4*. The proofs are relegated to *Section 5*.

1.1 Literature

There are two strands of theoretical literature related to our paper. The first follows the seminal papers by Alchian and Demsetz (1972) and Holmström (1982) to discuss the design of incentives in partnerships where the partners' contributions are not observable (and thus, contribution-sensitive sharing schemes like proportionality are not available). In contrast to Alchian and Demsetz (1972) who argue that efficiency can only be restored by bringing in a principle who monitors the agents, Holmström (1982) shows that group incentives can remove the free-rider problem.[4] The following literature focuses on the same question un-

[4] While we work under different informational assumptions, Holmström's question is similar to this paper. Quoting (pg 326): "The question is whether there is a way of fully allocating the joint outcome so that the resulting noncooperative game among the agents has a Pareto optimal Nash equilibrium." Holmström shows that the free rider problem can be solved as follows. One sets an output objective (by utilizing

der alternative assumptions. Kandel and Lazear (1992) analyze the effect of peer pressure, Legros and Mathews (1993) analyze the effect of limited liability, Miller (1997) and Strausz (1999) analyze cases where a partner can observe the effort exerted by a subset of other partners, and Morrison and Wilhelm (2004) discuss moral hazard problems associated with intergenerational transfer of human capital. Hart and Holmström (2010) and Hart (2011) adopt the "contracts as reference points" approach to discuss shading and efficient partnership contracts. Farrell and Scotchmer (1988) analyze the efficiency costs of equal-sharing in a theoretical model of partnership formation.

The studies in this literature focus on a stylized asymmetric information environment. Some of them casually mention that, if information asymmetry is not an issue, using the piece-rate (*i.e.* proportional) method would solve incentive problems (*e.g.* see Gaynor and Pauly, 1990). However, we are not aware of any formal study that models such an environment with possibility of both positive and negative surplus and that compares the incentive implications of partnership agreements which are commonly used in real life, as we do in this study. Maybe surprisingly, our results show that it is possible to improve upon the piece-rate method if one wishes to maximize the partners' total contributions. Additionally, a move away from proportionality (towards equal shares) can in turn lead to an increase in a partner's contribution. The above literature is also silent on the implications of using two separate methods to allocate positive and negative surplus. Our results show that this possibility (which exists in real life) has significant implications for partnerships.

The second strand of theoretical literature related to our paper is on axiomatic resource allocation. The partnership agreements that we consider are based on two principles (proportional versus equal sharing) central in the surplus sharing literature. See O'Neill (1982), Aumann and Maschler (1985) and the following literature (reviewed in Thomson 2003 and 2008) for axiomatic studies on allocating negative surplus (referred to as claims or bankruptcy problems by this literature). On the other hand, Moulin (1987) and the following literature provides an axiomatic study for positive surplus. There also is a smaller literature that covers both cases simultaneously. For example, Chun (1988) proposes characterizations of classes of rules that mix the proportionality and equal awards principles in both cases of positive and negative surplus. Herrero, Maschler, Villar (1999) propose and analyze a

the observable information about the agents' costs of effort). If it is not met, all partners receive zero as punishment. Otherwise, they share the produced value.

"rights-egalitarian solution" which uses the equal awards principle in case of positive surplus and the equal losses principle in case of negative surplus.

The axiomatic literature analyzes a much larger class of rules in comparison to the one following Holmtsröm (1982). However, studies in this literature focus on normative questions and typically remain silent on strategic issues, particularly the role of incentives in the formation of surplus. By focusing on this latter question and by analyzing the structure of productively and socially efficient partnership contracts, our paper contributes to this literature.

Some of our modeling choices are related to the previous literature as follows. First, there are many earlier papers that, like us, model the output as stochastic. For example, see Huddart and Liang (2003), Comino, Nicolo, Tedeschi (2010). Again similar to us, several earlier studies argue that the partners' expectations on their shares in case the partnership fails will have an effect on the partners' effort choices. For example, see Comino, Nicolo, Tedeschi (2010) or Li and Wolfstetter (2010). Finally, almost all the theoretical literature following Holmström (1982) uses additively separable utility functions (quasilinear preferences). Similar to those studies, we measure contributions in monetary units. But we alternatively assume that the agents have constant absolute risk aversion ($CARA$) utilities. Since we consider a stochastic production function, the CARA family provides us a good way to measure the affect of the agents' risk attitudes on the outcome. Analysis of a simple quasilinear model confirms our findings but turns out to be less interesting in terms of the interaction among the agents.

Finally it is useful to mention Kıbrıs and Kıbrıs (2013), where we use a similar modeling approach to analyze the investment implications of bankruptcy laws. While the two studies consider two separate economic institutions and contribute to two distinct strands of literature, they both analyze the incentive implications of resource allocation mechanisms in an environment with uncertainty and, in that sense, can be related to each other. In terms of this relation, it is useful to note that this paper analyzes a more complicated problem than Kıbrıs and Kıbrıs (2013). In that study, the allocation problem was restricted only to the "bad outcome" (in that case, bankruptcy) whereas here, it concerns both outcomes. Thus, the class of rules considered in Kıbrıs and Kıbrıs (2013) can be considered as a one-dimensional subset of the two-dimensional space of rules considered in this study. This increase in dimension both complicates and enriches the current analysis. For example, as will be detailed below, we show that the interaction between the cases of positive and negative surplus produces a number of surprising hypotheses regarding the design of partnership agreements.

2 Model

The set of **partners** is $N = \{1, ..., n\}$. Each partner $i \in N$ has the following **Constant Absolute Risk Aversion (CARA) utility function** $u_i : \mathbb{R} \to \mathbb{R}$ on money: $u_i(x) = -e^{-a_i x}$. Assume that each partner i is risk averse, that is, $a_i > 0$. Also assume that $a_1 \leq ... \leq a_n$.

Each partner i chooses his contribution to the partnership, $s_i \in \mathbb{R}_+$. We measure contributions in monetary units (or equivalently assume a constant marginal cost normalized to 1). The total contribution of the partners is then $\sum_N s_j$. With **success probability** $p \in (0, 1)$, this value brings a **return** $r \in (0, 1]$ and becomes $(1+r) \sum_N s_j$, creating a **positive surplus** of $r \sum_N s_j$ for the partners. With the remaining $(1 - p)$ probability, the partnership's value becomes $\beta \sum_N s_i$ where $\beta \in (0, 1)$ is the **fraction that survives failure.** In this case, the partnership makes a **negative surplus** of $(1 - \beta) \sum_N s_i$.

A **partnership agreement** is a pair of rules F, G to be used in case of positive and negative surplus, respectively. The **positive-surplus rule** F allocates the gross returns $(1 + r) \sum s_j$ according to the vector of contributions s, partner i's share being $F_i(s, (1 + r) \sum s_j)$. The **negative-surplus rule** G, on the other hand, allocates the amount that survives failure $\beta \sum s_j$ according to the vector of contributions s, partner i's share being denoted as $G_i(s, \beta \sum s_j)$.

The following partnership agreements are based on two central surplus-sharing rules commonly used in real life. Suppose the partnership creates value V. (From previous discussion, we know V is either $(1 + r) \sum s_j$ or $\beta \sum s_j$. But the next two definitions will be independent of what V is.) The **proportional surplus-sharing rule**, P, allocates the surplus proportional to the partners' contributions. The share of a typical agent is then $P_i (s, V) = \frac{s_i}{\sum s_j} V = s_i + \frac{s_i}{\sum s_j} (V - \sum s_j)$ (where $V - \sum s_j$ is the surplus). The **equal surplus-sharing rule**, E, allocates the surplus equally. The share of an agent is then $E_i (s, V) = s_i + \frac{V - \sum s_j}{n}$.

Gaynor and Pauly (1990) mention that the following "mixtures" of P and E are also commonly used, especially in professional service partnerships. For each $\rho \in [0,1]$, the **PE $[\boldsymbol{\rho}]$ rule** first reimburses each partner for his contributions. Then, it allocates $(1-\rho)$ part of the surplus equally among the partners and uses the remaining fraction ρ to give bonuses in proportion to contributions:

$$\begin{aligned} PE[\rho]_i\,(s,V) &= \rho P_i\,(s,V) + (1-\rho)\,E_i\,(s,V) \\ &= s_i + \left(V - \sum s_j\right)\left(\rho \frac{s_i}{\sum s_j} + (1-\rho)\,\frac{1}{n}\right). \end{aligned}$$

Geometrically, these rules span all convex combinations of the proportional and equal surplus-share allocations.

As noted in the introduction, a partnership agreement can specify different rules to be used in cases of positive and negative surplus. The class of partnership agreements that we analyze, therefore combine a positive-surplus rule $PE\,[\gamma]$ and a negative surplus rule $PE\,[\alpha]$ where $\alpha, \gamma \in [0,1]$ and $\alpha \neq \gamma$ is allowed. We will refer to a **partnership agreement** as $PE\,[\gamma, \alpha]$.

Given the partnership agreement $PE\,[\gamma, \alpha]$, the partners simultaneously choose their contributions. Agent i's (expected) payoff from a strategy (*i.e.* contribution) profile $s \in \mathbb{R}^n_+$ is

$$U_i^{PE[\gamma,\alpha]}(s) = pu_i\left(F_i\left(s, (1+r)\sum s_j\right) - s_i\right) + (1-p)u_i\left(G_i\left(s, \beta \sum s_j\right) - s_i\right)$$

where $F_i\,(s, (1+r)\sum s_j) - s_i$ and $G_i\,(s, \beta \sum s_j) - s_i$ are his surplus shares in cases of positive and negative surplus, respectively. Let $U^{PE[\gamma,\alpha]} = \left(U_1^{PE[\gamma,\alpha]}, ..., U_n^{PE[\gamma,\alpha]}\right)$. The **partnership game induced by** $PE\,[\gamma, \alpha]$ is then defined as

$$\mathcal{G}^{PE[\gamma,\alpha]} = \langle \mathbb{R}^N_+, U^{PE[\gamma,\alpha]} \rangle.$$

Let $\epsilon(\mathcal{G}^{PE[\gamma,\alpha]})$ denote the set of Nash equilibria of $\mathcal{G}^{PE[\gamma,\alpha]}$.

To measure the partners' social welfare from a partnership agreement, we will resort to two leading measures in the literature. The **egalitarian social welfare induced by** $PE\,[\gamma, \alpha]$ is the minimum utility an agent obtains at the Nash equilibrium of the partnership game induced by $PE\,[\gamma, \alpha]$:

$$\mathcal{EG}^{PE[\gamma,\alpha]}\,(p, r, \beta, a_1, ..., a_n) = \min_{i \in N} U_i(\epsilon\left(G^{PE[\gamma,\alpha]}\right)).$$

The **utilitarian social welfare induced by** $PE[\gamma,\alpha]$ is the total utility the agents obtain at the Nash equilibrium of the partnership game induced by $PE[\gamma,\alpha]$:

$$\mathcal{UT}^{PE[\gamma,\alpha]}(p,r,\beta,a_1,...,a_n)=\sum_{i\in N}U_i(\epsilon\left(G^{PE[\gamma,\alpha]}\right)).$$

3 Results

As defined in the previous section, each partnership agreement $PE[\gamma,\alpha]$ induces a partnership game among the agents. We next analyze the Nash equilibria of these games to discuss partnership formation, equilibrium contributions and productive as well as social efficiency.

3.1 Partnership Formation: Acceptable Agreements

In this section, we analyze the conditions under which the agents in N are going to form a partnership. We argue that if a group of agents form a partnership, the expectation is that all will contribute to it (though probably at different levels). However, if the common expectation is that some agents will *contribute nothing* to the partnership, the agreement $PE[\gamma,\alpha]$ will not be acceptable for the contributing partners and the partnership will not form in the first place.

Formally, we say that a partnership agreement $PE[\gamma,\alpha]$ is **acceptable** for N if at the Nash equilibrium of the partnership game, all partners choose a positive contribution. Otherwise, a group of agents choose not to participate (by choosing zero contributions) and the partnership does not form. [5]

Two conditions turn out to be important in determining whether the agreement $PE[\gamma,\alpha]$ is acceptable for N. The first condition, **profitability**, requires:

$$\ln\left(\frac{pr\,(n\gamma-\gamma+1)}{(1-p)\,(1-\beta)\,(n\alpha-\alpha+1)}\right)>0. \qquad \text{(Profitability)}$$

This condition, which can be rewritten as $pr\,(n\gamma-\gamma+1)>(1-p)\,(1-\beta)\,(n\alpha-\alpha+1)$, simply compares the return on unit contribution in case of positive surplus, $r\,(n\gamma-\gamma+1)$, weighted by the probability of success, p, with the loss incurred on unit contribution in case

[5] As will be discussed in detail later, under (and only under) $PE[0,0]$, the partnership game will have a continuum of Nash equilibria when $a_1=...=a_n$. For this case, we will say that $PE[0,0]$ is acceptable if there is at least one Nash equilibrium where all partners choose positive contributions.

of negative surplus, $(1-\beta)(n\alpha-\alpha+1)$, weighted by the probability of failure, $(1-p)$. Positive contributions are optimal if the returns in case of success outweigh the losses incurred in case of failure.[6] Note that the *Profitability condition* does not make any reference to the partners' risk attitudes. That will be the concern of our next condition.

The second condition, **homogeneity,** requires that the agents are not too different in terms of their risk attitudes :

$$\frac{\frac{1}{a_n}}{\frac{1}{n}\left(\sum_N \frac{1}{a_j}\right)} > 1 - \frac{\gamma r + \alpha(1-\beta)}{r+1-\beta}. \tag{Homogeneity}$$

The left hand side of this inequality has played an important role in previous studies such as Wilson (1968) and Huddart and Liang (2003). It is interpreted as agent n's *risk tolerance* relative to the average risk tolerance of the partnership (*e.g.* see Wilson's interpretation for the case of syndicates). Since agent n is the most risk averse partner (i.e. $a_1 \leq ... \leq a_n$), the left hand side is less than or equal to 1 (and it is equal to 1 precisely when $a_1 = ... = a_n$). For the same reason, if agent n were to be replaced with any other agent, the left hand side would increase in value, making the inequality less binding. This is why the *Homogeneity condition* is stated for agent n, even though it applies to all partners.

The right hand side of the inequality depends on how distant $PE[\gamma,\alpha]$ is from pure proportionality, $PE[1,1]$. The denominator of the fraction shows how $PE[1,1]$ allocates positive surplus (r) and negative surplus $(1-\beta)$. The nominator, on the other hand, shows that under $PE[\gamma,\alpha]$, only γ fraction of positive surplus and α fraction of negative surplus is allocated proportionally (γr and $\alpha(1-\beta)$). When both α and γ are 1, that is for $PE[1,1]$, the right hand side is zero and thus, not binding. As either of the two surplus sharing rules move towards equal shares however, that is, as α or γ goes down, the right hand side increases, becoming more binding. When $\alpha = \gamma = 0$ (*i.e.* when the partnership agreement allocates

[6] To see this, divide both sides by n and obtain $p\left(\gamma r + (1-\gamma)\frac{r}{n}\right) > (1-p)\left(\alpha(1-\beta) + (1-\alpha)\frac{(1-\beta)}{n}\right)$. The left hand side expression $\left(\gamma r + (1-\gamma)\frac{r}{n}\right)$ has two parts. The γ weighted part r is the partner's return under proportional surplus-sharing and the $(1-\gamma)$ weighted part $\frac{r}{n}$ is his return under equal surplus-sharing. Thus, the weighted average $\left(\gamma r + (1-\gamma)\frac{r}{n}\right)$ is the partner's return under the positive-surplus rule $PE[\gamma]$. The right hand side expression $\left(\alpha(1-\beta) + (1-\alpha)\frac{(1-\beta)}{n}\right)$ again has two parts. The α weighted part of this expression, $(1-\beta)$ is the loss incurred for unit effort in case of proportional surplus-sharing and the $(1-\alpha)$ weighted part $\frac{1-\beta}{n}$ is the loss incurred in case of equal surplus-sharing. Thus, the weighted average $\left(\alpha(1-\beta) + (1-\alpha)\frac{(1-\beta)}{n}\right)$ is the partner's losses under the negative-surplus rule $PE[\alpha]$.

both positive and negative surplus equally), the right hand side reaches its maximum value of 1.

Proposition 1 ***(Partnership formation under*** $PE[\gamma,\alpha]$***)*** *A partnership agreement* $PE[\gamma,\alpha]$ *with* $max\{\alpha,\gamma\} > 0$ *is acceptable for* N *if and only if both Profitability and* Homogeneity *conditions are satisfied.*
The partnership agreement $PE[0,0]$ *is acceptable for* N *if and only if Profitability is satisfied and the* Homogeneity *condition holds with a weak inequality.*

Note that when $\alpha = \gamma = 0$, the right hand side of the *Homogeneity condition* is 1. The maximum value for the left hand side, achieved when $a_1 = ... = a_n$, is also 1. Thus, when $\alpha = \gamma = 0$, the *Homogeneity condition* holds with a weak inequality if and only if all agents have identical risk attitudes. This is precisely the case when the partnership game has multiple Nash equilibria and for that reason, it will require special attention, as can be seen below.

3.2 Equilibrium Contributions

In this section, we analyze the equilibrium contributions of partners in a formed partnership. As can be seen in the following proposition, equilibrium contributions are unique under all partnership agreements but $PE[0,0]$.

Proposition 2 ***(Equilibrium contributions under*** $PE\left[\gamma,\alpha\right]$***)*** *If the agreement* $PE[\gamma,\alpha]$ *with* $max\{\alpha,\gamma\} > 0$ *is acceptable for* N, *the resulting partnership game has a unique Nash equilibrium* s^* *where*

$$s_i^* = \frac{\left(n\left(r+1-\beta\right)\frac{1}{a_i} - \left(\left(1-\gamma\right)r + \left(1-\alpha\right)\left(1-\beta\right)\right)\left(\sum_N \frac{1}{a_j}\right)\right)\ln\left(\frac{pr(n\gamma-\gamma+1)}{(1-p)(1-\beta)(n\alpha-\alpha+1)}\right)}{n\left(r+1-\beta\right)\left(\gamma r+\alpha\left(1-\beta\right)\right)} \tag{1}$$

for each $i \in N$.
On the other hand, if $PE\left[0,0\right]$ *is acceptable for* N, *the partnership game has a continuum of Nash equilibria: any contribution profile* $s^* \geq 0$ *such that*

$$\sum_N s_i^* = \frac{n\ln\left(\frac{pr}{(1-p)(1-\beta)}\right)}{a_n\left(1-\beta+r\right)}$$

is a Nash equilibrium.

Note that the ln term in Equation (1) is the one used in the *Profitability condition.* Also, as can easily be checked, the denominator of the first term in Equation (1) is always positive. The *Homogeneity condition* guarantees that the nominator is of positive sign a well.

As stated in Proposition 1, under $PE\,[0,0]$ a partnership forms if only if $a_1 = ... = a_n$. Proposition 2 then tells us that this symmetric game has a continuum of Nash equilibria. Nevertheless, the symmetric equilibrium among them (where for each $i \in N$, $s_i^* = \frac{\ln\left(\frac{pr}{(1-p)(1-\beta)}\right)}{a_i(1-\beta+r)}$) is more desirable than the rest in the following sense. Imagine a sequence of partnership agreements, each satisfying $max\{\alpha,\gamma\} > 0$, but converging to $PE\,[0,0]$. As can be seen from Proposition 2, the corresponding sequence of unique equilibrium contributions will also be converging, and it will converge precisely to this symmetric equilibrium under $PE\,[0,0]$. No other equilibrium under $PE\,[0,0]$ satisfies this property. Therefore, in welfare comparisons, we will focus on this symmetric equilibrium under $PE\,[0,0]$ and $a_1 = .. = a_n$.

Since $a_1 \leq ... \leq a_n$, *Equation 1* implies $s_1^* \geq ... \geq s_n^*$. That is, agent i is a "bigger partner" than agent j whenever $i \leq j$.

A corollary of Proposition 2 identifies conditions under which the partnership game has a dominant strategy equilibrium.[7] Partnership agreements that induce dominant strategy equilibria are advantageous to those that do not since it is possible to make a stronger prediction about how the partners will behave.

Corollary 3 ***(Dominant strategy equilibrium under*** $PE\,[1,1]$***)*** *The partnership game induced by the agreement* $PE\,[1,1]$ *has a dominant strategy equilibrium (in strictly dominant strategies). No other partnership agreement induces dominant strategy equilibria.*

To provide the reader with a better understanding of the above propositions, we conclude this section with a numerical example that demonstrates how individual contributions depend on the partnership agreement $PE\,[\gamma,\alpha]$. In the example, the parameter values are $r = 0.3$, $p = 0.8$, $\beta = 0.7$, $a_1 = 1$, $a_2 = 1.5$, $\gamma = 0.5$.

Figures 1 and 2 plot how individual contributions change as a function of α, the percentage of negative surplus allocated proportionally. As can be seen in Figure 1, an increase in α decreases Partner 1's contribution. This might seem surprising at first glance, since it is

[7]It follows from Equation (1) that the partnership games induced by $PE\,[\gamma,\alpha]$ agreements admit dominant strategy equilibria if and only if $(1-\gamma)\,r + (1-\alpha)\,(1-\beta) = 0$ (in which case partner i's best response is independent of the other agents' contributions). This equality holds if and only if $\alpha = \gamma = 1$.

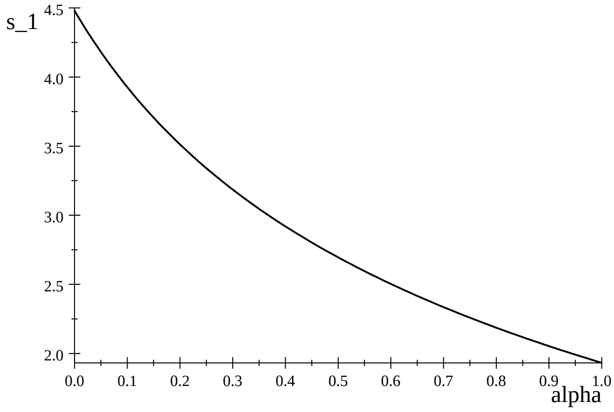

Figure 1: Partner 1's equilibrium contributions, as a function of α.

commonly argued in the literature that a shift from equal to proportional surplus shares will increase individual contributions. However, the reader will note after a closer inspection that an increase in α decreases the marginal return on contributions in case of negative surplus (by making losses more sensitive to contributions). It thereby induces both partners to contribute less.

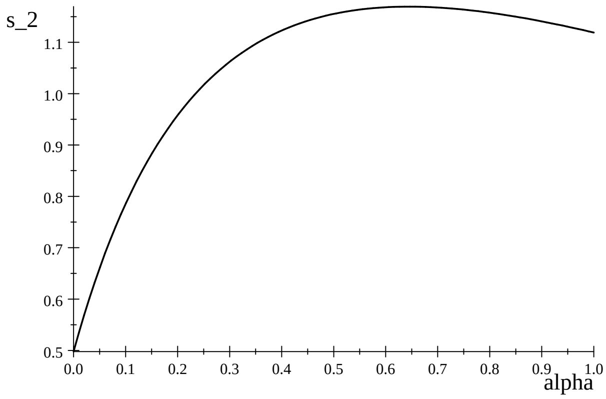

Figure 2: Partner 2's equilibrium contributions, as a function of α.

Maybe more surprisingly, Figure 2 shows that α has a non-monotonic effect on the contribution of the smaller partner, Partner 2, who first increases and then decreases his contribution. This nonmonotonicity is caused by two competing effects. The first, direct effect is already mentioned in the previous paragraph. The second, indirect effect is due to the fact that the two partners' contributions are strategic substitutes. Thus, as Partner 1 decreases

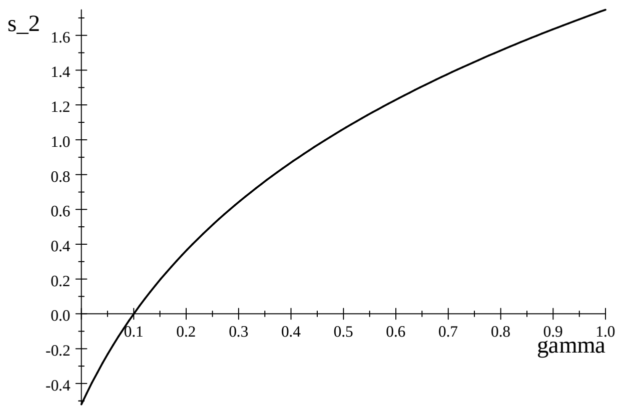

Figure 3: Partner 2's equilibrium contributions, as a function of γ.

his contribution in response to an increase in α, partner 2 is inclined to increase his own contribution in response. The figure shows that the latter affect is dominant for small values of α. But for high α values, the first direct effect overtakes the second.[8]

The nonmonotonicity of s_2^* in α is not a knife-edge case. In this example, unilateral changes in γ or r do not disturb this nonmonotonicity; a unilateral change in p disturbs it only when $p > 0.87$ (making s_2^* an increasing function) and a unilateral change in β disturbs it only when $\beta > 0.95$ (making s_2^* a decreasing function). It is also useful to note that, for the above parameter values, the value of α that maximizes s_2 is decreasing in γ (the percentage of positive surplus allocated proportionally). This shows that the incentives Partner 2 faces are not straightforward, but are determined through and interplay of the positive-surplus and negative-surplus rules.

In the same example, we next fix $\alpha = 0.3$ and let γ vary. Figure 3 demonstrates that, as claimed by the previous literature, an increase in γ (the percentage of positive surplus allocated proportionally) in turn increases Partner 2's contributions.[9] However, as shown in Figure 4, the effect of γ on Partner 1 is non-monotonic. (The discussion, similar to the case of α, is omitted.) Thus, contrary to what the previous literature suggests, moving from a

[8]More formally, both partners have linear best response functions (with a positive intercept and a negative slope). An increase in α affects both best response functions in the same way: it decreases the intercept and decreases the slope in absolute value, making it less sensitive to the other partner's choices. It is because of this that the strategic substitutes property matters less at high values of α.

[9]Figure 3 also demonstrates that, for $\gamma \leq 0.1$, the partnership agreement $PE[\gamma, \alpha]$ is not acceptable and, as discussed in the previous section, the partnership does not form.

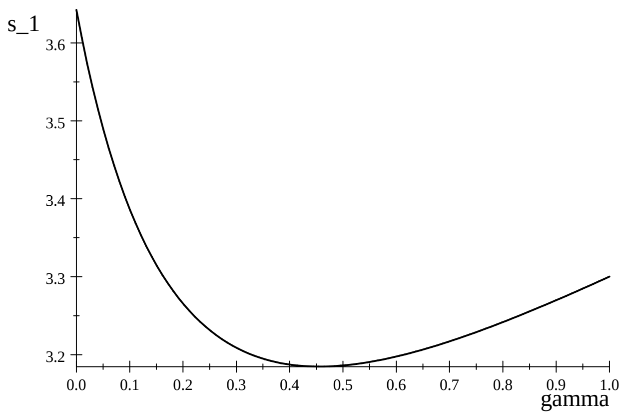

Figure 4: Partner 1's equilibrium contributions, as a function of γ.

fixed surplus-sharing rule towards proportional shares (the piece-rate) does not necessarily increase individual contribution for all partners.

3.3 Productive Efficiency

In this section, we compare partnership agreements in terms of the total contribution that they induce in equilibrium, that is, in terms of their *productive efficiency*. As demonstrated in the previous section, a look at individual contributions suggests no clear prediction as to how total contributions would be affected from changes in the underlying partnership agreement. On the other hand, figures 5 and 6 suggest a clear ordering in our numerical example. First, the choice of γ affects total contribution as in Figure 5. This figure confirms the common belief that a move from equal surplus-sharing towards proportionality increases total contributions. Figure 6, however, shows that a similar move in the allocation of negative surplus has the opposite effect.

The following theorem shows that what we observe in this numerical example in fact generalizes to the whole parameter space.

Theorem 1 *Under the $PE[\gamma, \alpha]$ family of agreements, equilibrium total contribution is (i) increasing in γ (the percentage of positive surplus allocated proportionally) and (ii) decreasing in α (the percentage of negative surplus allocated proportionally). Furthermore, both effects are increasing in the number of partners in the partnership.*

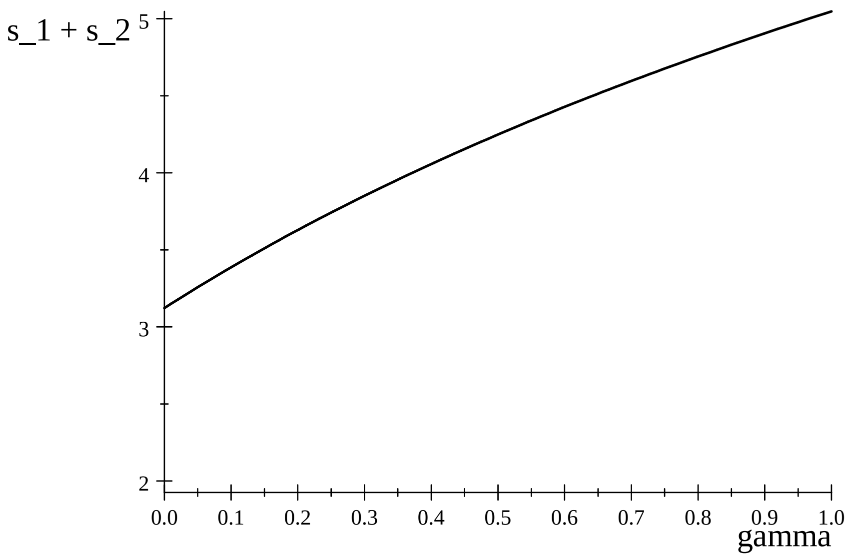

Figure 5: The effect of γ on total effort. The parameters are $r = 0.3$, $p = 0.8$, $\beta = 0.7$, $a_1 = 1$, $a_2 = 1.5$, $\alpha = 0.3$.

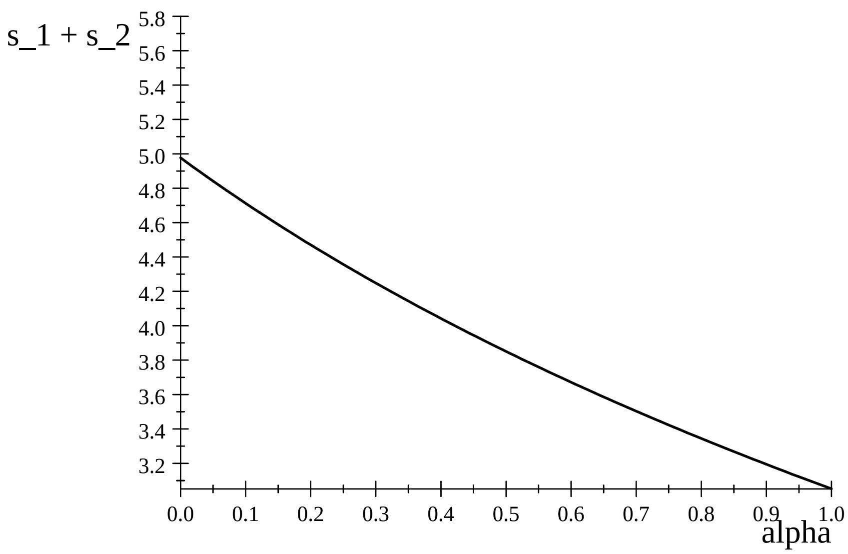

Figure 6: The effect of α on total effort. The parameters are $r = 0.3$, $p = 0.8$, $\beta = 0.7$, $a_1 = 1$, $a_2 = 1.5$, $\gamma = 0.5$.

In terms of what it says regarding the positive-surplus rule, the theorem supports the general view that moving from a fixed surplus-sharing rule (like E) to proportional division increases the total contributions the partners put into the partnership. For the negative-surplus rule, however, the theorem identifies that now, a move away from proportional division towards equal surplus-sharing increases the total contributions the partners put into the partnership.

The theorem, thus, shows us that a way to improve over the commonly-used piece rate agreement is to change the surplus-sharing rule used in case of negative-surplus; a move towards equal surplus shares helps productive efficiency. While such a change does not incentivize every partner to contribute more (*e.g.* see Partner 2 in Figure 2), it incentivizes the bigger (less risk averse) partners who now transfer less to the smaller partners in case of negative surplus. Additionally, the bigger partners respond more to an increase in α than the smaller partners do, making the aggregate effect of α negative.

It is, however, interesting to note that, even in symmetric partnerships (*i.e.* when all partners have identical risk attitudes), the ordering of partnership agreements in terms of total contributions is still as above. Particularly, $PE\,[1,0]$ still remains as the unique productively efficient agreement. It is also important to reiterate that the effect of the agreement on total contributions is emphasized in partnerships with a greater number of partners. Thus, one would expect bigger partnerships to pick greater γ and smaller α parameters.

Theorem 1 implies that the partnership agreement $PE\,[1,0]$ is the unique productively efficient agreement in the $PE\,[\gamma,\alpha]$ family. However, as discussed in Subsection 3.1, there are partnerships where this agreement will not be acceptable. In such partnerships, $PE\,[1,0]$ violates either the *Profitability* or the *Homogeneity* condition. First, it is straightforward to see that if $PE\,[1,0]$ violates *Profitability*, every other partnership agreement also does so. Thus, in such cases the partnership will not form under any $PE\,[\gamma,\alpha]$ agreement. The more interesting case is when $PE\,[1,0]$ violates *Homogeneity*. Then, an increase in α helps to satisfy the inequality while a decrease in γ does not. Thus, keeping $\gamma = 1$, there is a critical value

$$\alpha^{*} = 1 - \frac{r+1-\beta}{1-\beta}\frac{\frac{1}{a_n}}{\sum_N \frac{1}{a_j}}$$

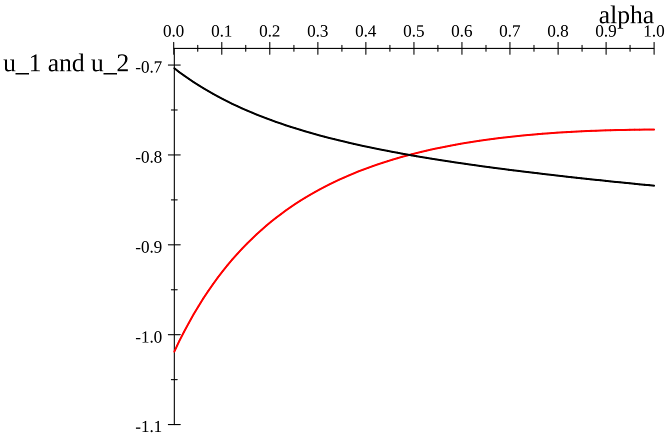

Figure 7: Utility of Partner 1 (black) and Partner 2 (red) as a function of α. The weight of the success rule $\gamma = 0.5$.

where for each $\alpha \leq \alpha^*$, $PE\,[1, \alpha]$ is not acceptable.[10] The set of acceptable agreements are thus $PE\,[1, \alpha]$ such that $\alpha > \alpha^*$. As α decreases, productive efficiency increases and simultaneously the most risk averse Partner n's contribution decreases. At the limit $\alpha = \alpha^*$, Partner n picks a zero contribution making $PE\,[1, \alpha^*]$ unacceptable.

3.4 Individual and Social Welfare

In this section, we look at the individual and social welfare levels induced by alternative partnership agreements. We make an analytical comparison in terms of egalitarian social welfare. Additionally, we carry out a numerical analysis in terms of utilitarian social welfare.

Figures 7 and 8 demonstrate how equilibrium welfare of the two partners in our example change as α and γ change in their $PE\,[\gamma, \alpha]$ partnership agreement.

[10]To see this, note that

$$
\begin{aligned}
\frac{\frac{1}{a_n}}{\sum_N \frac{1}{a_j}} &> 1 - \frac{r + \alpha\,(1-\beta)}{r + 1 - \beta} \text{ iff} \\
\alpha &> \frac{(r + 1 - \beta)}{(1-\beta)} \left(1 - \frac{\frac{1}{a_n}}{\sum_N \frac{1}{a_j}}\right) - \frac{r}{(1-\beta)} \text{ iff} \\
\alpha &> 1 - \frac{r + 1 - \beta}{1-\beta} \frac{\frac{1}{a_n}}{\sum_N \frac{1}{a_j}}
\end{aligned}
$$

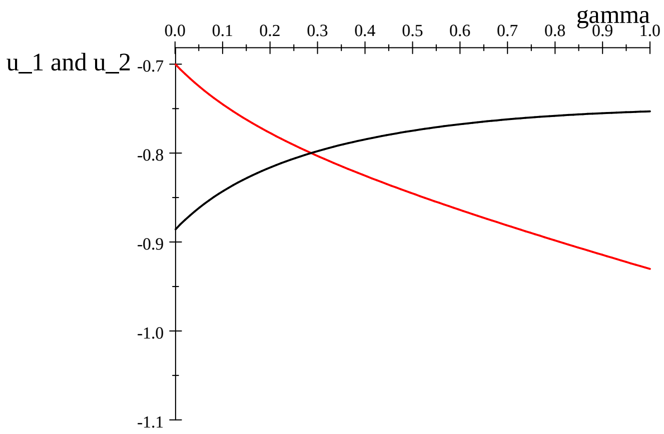

Figure 8: Utility of Partner 1 (black) and Partner 2 (red) as a function of γ. The weight of the failure rule $\alpha = 0.3$.

The following observations are in order. First, in both pictures Partner 1 (the bigger partner) receives a greater utility than Partner 2 if and only if $\alpha < \gamma$, that is, when the positive-surplus rule is closer to proportionality than the negative-surplus rule. This means that in this example, egalitarian social welfare is equal to the utility of Partner 1 when $\alpha > \gamma$ and to the utility of Partner 2 when $\alpha < \gamma$. As both pictures demonstrate, when $\alpha = \gamma$, the two partners receive equal payoff. Second, this egalitarian social welfare increases as α and γ gets closer to each other, and is maximized at $\alpha = \gamma$.

Surprisingly, both of the above points are generalizable to an arbitrary number of agents and to all parameter values we consider. The following proposition orders the agents according to their equilibrium welfare.

Proposition 4 *Under the $PE\,[\gamma, \alpha]$ family of partnership agreements, the partners are ordered according to their equilibrium utilities as* $1, 2, ..., n$. *If $\alpha > \gamma$, the least risk-averse Partner* 1 *always receives the smallest utility and the most risk-averse Partner n always receives the highest utility. If $\alpha < \gamma$, the ordering is reversed, Partner* 1 *now receiving the highest utility and Partner n, the smallest. Finally, if $\alpha = \gamma$, all partners receive the same utility level.*

The above proposition implies that the egalitarian social welfare is equal to the equilibrium payoff of either the most or the least risk averse partner, depending on the α-γ

relationship in their partnership agreement. The following theorem shows that this egalitarian social welfare is maximized at $\alpha = \gamma$.

Theorem 2 *Under the $PE[\gamma, \alpha]$ family of partnership agreements, egalitarian social welfare is decreasing in $|\alpha - \gamma|$. When $\alpha = \gamma = x$, all partners' payoffs are equal and this common payoff, which is also the egalitarian social welfare under the $PE[x, x]$ partnership agreement, is independent of x.*

While all $PE[x, x]$ partnership agreements induce the same egalitarian social welfare level, they might be different in other aspects. The first that comes to mind is the agents' contribution choices. It turns out that all $PE[x, x]$ partnership agreements induce the same total contribution in equilibrium. These agreements, however, differ in terms of the individual contributions that they induce in equilibrium. Partners who are less (more) risk averse than the average decrease (increase) their contributions in response to an increase in the common x, keeping total contributions constant. (For a proof, please see Claim 1 in Section 5.)

Due to differences in individual contribution choices, it might be that some $PE[x, x]$ agreements are acceptable while the others are not (as discussed in Subsection 3.1). It is straightforward to check that the *Profitability* condition does not distinguish among the $PE[x, x]$ agreements; either they all satisfy or violate it. The *Homogeneity condition*, on the other hand, partitions the set of $PE[x, x]$ agreements. There is a critical value

$$x^* = 1 - \frac{\frac{1}{a_n}}{\sum_N \frac{1}{a_j}}$$

where an agreement $PE[x, x]$ is acceptable if and only if $x > x^*$. As the common x decreases in an acceptable agreement, the most risk averse Partner n's contribution will also decrease, reaching zero at $x = x^*$.

We conclude this section with a discussion of utilitarian social welfare. For this case, the ordering of partnership agreements in terms of utilitarian social welfare depends on the underlying parameter values. This makes a general analytical result as in the case of egalitarian social welfare infeasible. However, a numerical analysis shows that utilitarian social welfare ordering of agreements is not radically different than egalitarian social welfare ordering.

We first carry out a numerical analysis for the case of two partners. We allow the following parameter values:

$$\begin{aligned} \beta, p, r, \alpha, \gamma &\in \{0.01, 0.1, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.99\}, \\ a_1, a_2 &\in \{0.1, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5\} \text{ and } a_2 \geq a_1. \end{aligned}$$

After eliminating parameter combinations that violate the Profitability and Homogeneity conditions, we end up with 1 107 936 parameter combinations. Surprisingly, at 1 082 387 (that is, at 97.7 %) of these parameter combinations, utilitarian social welfare is maximized when $\gamma = \alpha$. At the remaining 25 549 parameter combinations, utilitarian social welfare is maximized at a $\gamma \neq \alpha$ and the two cases $\gamma > \alpha$ and $\gamma < \alpha$ are observed at almost equal frequency. At 5 758 (that is, 22.5 %) of this 25 549, the difference between γ and α is one grid point.

We also carried out a numerical analysis for the case of three partners. Since the computer could not handle the above grid, we switched to a slightly coarser grid of

$$\begin{aligned} \beta, p, r &\in \{0.01, 0.16, 0.31, 0.46, 0.51, 0.66, 0.71, 0.86, 0.91\}, \\ \alpha, \gamma &\in \{0.01, 0.1, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.99\}, \\ a_1 &\in \{0.1, 0.7, 1.3, 1.9, 2.5, 3.1, 3.7, 4.3, 4.9\} \text{ and } a_3 \geq a_2 \geq a_1. \end{aligned}$$

After eliminating parameter combinations that violate the Profitability and Homogeneity conditions, we end up with 1 625 934 parameter combinations. Similar to the two-partner case, at 1 607 475 (that is, at 98.9 %) of these parameter combinations, utilitarian social welfare is maximized when $\gamma = \alpha$. At the remaining 18 459 parameter combinations, utilitarian social welfare is maximized at a $\gamma \neq \alpha$; at 6 954 of which $\gamma > \alpha$ and at 11 505 of which, $\gamma < \alpha$. At 5 245 (that is, 28.4 %) of this 18 459, the difference between γ and α is 1.

These numerical findings should be interpreted with caution. The $\gamma = \alpha$ finding, in a significant number of the cases, is due to the grid that we impose on the parameter space. Thus, we can only deduce from this analysis that in most cases, utilitarian social welfare is maximized at α, γ values that are close to each other and that, maximizing utilitarian social welfare does not create agreements that are radically different than those that maximize egalitarian social welfare.

4 Conclusion

Our analysis compares a family of partnership agreements (i.e. surplus allocation rules) in terms of total contributions and social welfare that they induce in equilibrium of a noncooperative partnership game. Our findings are as follows:

(*i*) Independent of the parameter values considered, equilibrium total contributions induced by a partnership agreement increases as the positive-surplus rule gets closer to proportionality and the negative-surplus rule gets closer to equal surplus-shares. Using proportionality in case of positive surplus and equal-surplus shares in case of negative surplus maximizes total contributions.

(*ii*) Independent of the parameter values considered, egalitarian social welfare increases as the percentages of positive and negative surplus allocated proportionality (i.e. γ and α) get closer to each other. Partnership agreements where $\gamma = \alpha$ all maximize egalitarian social welfare. Such agreements give all agents the same welfare and produce the same amount of total contributions. They, however, are different in terms of individual contribution choices that they induce.

(*iii*) The ordering of partnership agreements in terms of utilitarian social welfare depends on the parameter values. Thus a general statement as in egalitarian social welfare or total contributions can not be made. However, a numerical analysis shows that the utilitarian optimal partnership agreements are not radically different than egalitarian optimal ones. Simulations for two and three agent partnerships show that at around 98 % of the parameter space, utilitarian social welfare is maximized when $\gamma = \alpha$.

(*iv*) In symmetric games (where $a_1 = ... = a_n$), the egalitarian optimal agreements described in (*ii*) additionally Pareto dominate all other agreements.

(*v*) There always is a unique dominant strategy equilibrium under the partnership agreement which uses proportionality when allocating both positive and negative surplus. No other partnership agreement induces dominant strategies.

Overall, we observe a trade-off between maximizing total contributions and social welfare.

To maximize total contributions, two opposite surplus allocation rules needs to be used in cases of positive and negative surplus. However, maximizing egalitarian social welfare (and as our numerical analysis reveals, utilitarian welfare to a certain extent) requires choosing the same surplus-allocation rule in cases of both positive and negative surplus. As noted in the Introduction, state partnership acts of many countries impose the same surplus-allocation rule for both positive and negative surplus. Thus, they seem to have picked the welfare side of this trade-off.

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5 Appendix

We will start this section by calculating the Nash equilibrium of the partnership game.

Under the family $PE\left[\gamma,\alpha\right]$, the positive-surplus rule is

$$PE\left[\gamma\right]_i\left(s,(1+r)\sum s_j\right) = s_i + \gamma r s_i + (1-\gamma)\frac{r\sum s_j}{n}.$$

and the negative-surplus rule is

$$PE[\alpha]_i\left(s,\beta\sum s_j\right) = \frac{n\alpha\beta + (n-1+\beta)(1-\alpha)}{n}s_i - \frac{(1-\alpha)(1-\beta)}{n}\sum_{N\setminus i} s_j.$$

Thus, the utility function of partner i is

$$U_i^{PE[\gamma,\alpha]}(s) = -pe^{-a_i\left(\gamma r s_i + (1-\gamma)\frac{r s_i + r\sum_{N\setminus i} s_j}{n}\right)} - (1-p)e^{\left(\frac{(1-\beta)(1+(n-1)\alpha)}{n}\right)a_i s_i + \frac{(1-\alpha)(1-\beta)}{n}a_i\sum_{N\setminus i} s_j}.$$

Its first derivative is

$$\begin{aligned}\frac{\partial U_i^{PE[\gamma,\alpha]}(s_i,s_{-i})}{\partial s_i} &= \frac{1}{n}pra_i\left(n\gamma-\gamma+1\right)\exp\left(-a_i\left(r\gamma s_i-\frac{1}{n}\left(rs_i+\left(\sum_{N\backslash i}s_j\right)r\right)(\gamma-1)\right)\right)\\ &\quad -\frac{1}{n}\exp\left(\frac{\left(\sum_{N\backslash i}s_j\right)}{n}a_i\left(\alpha-1\right)\left(\beta-1\right)-\frac{1}{n}a_is_i\left(\alpha\left(n-1\right)+1\right)\left(\beta-1\right)\right)\\ &\quad \times a_i\left(1-\beta\right)\left(1-p\right)\left(n\alpha-\alpha+1\right)\end{aligned}$$

and its second derivative is negative:

$$\begin{aligned}\frac{\partial^2 U_i^{LP}(s_i,s_{-i})}{\partial s_i^2} &= -\frac{1}{n^2}pr^2a_i^2\left(n\gamma-\gamma+1\right)^2\exp\left(-a_i\left(r\gamma s_i-\frac{1}{n}\left(rs_i+\left(\sum_{N\backslash i}s_j\right)r\right)(\gamma-1)\right)\right)\\ &\quad +\frac{1}{n^2}\exp\left(\frac{\left(\sum_{N\backslash i}s_j\right)}{n}a_i\left(\alpha-1\right)\left(\beta-1\right)-\frac{1}{n}a_is_i\left(\alpha\left(n-1\right)+1\right)\left(\beta-1\right)\right)\\ &\quad \times a_i^2\left(\beta-1\right)^2\left(p-1\right)\left(n\alpha-\alpha+1\right)^2\\ &< 0.\end{aligned}$$

Equating the first derivative to zero, we obtain as a function of s_{-i} :

$$\begin{aligned}s_i &= \sigma_i\left(s_{-i}\right) \qquad (2)\\ &= \frac{n\ln\left(\frac{pr(n\gamma-\gamma+1)}{(1-p)(1-\beta)(n\alpha-\alpha+1)}\right)}{a_i\left(\left(1-\alpha+n\alpha\right)\left(1-\beta\right)+\left(n-1\right)r\gamma+r\right)}-\frac{r\left(1-\gamma\right)+\left(1-\beta\right)\left(1-\alpha\right)}{\left(1-\alpha+n\alpha\right)\left(1-\beta\right)+\left(n-1\right)r\gamma+r}\left(\sum_{N\backslash i}s_j\right).\end{aligned}$$

Note that the slope of Expression 2 is negative since[11]

$$\frac{r(1-\gamma)+(1-\beta)(1-\alpha)}{(1-\alpha+n\alpha)(1-\beta)+(n-1)r\gamma+r}\in[0,1].$$

Also note that the sign of the constant term

$$\frac{n\ln\left(\frac{pr(n\gamma-\gamma+1)}{(1-p)(1-\beta)(n\alpha-\alpha+1)}\right)}{a_i\left((1-\alpha+n\alpha)(1-\beta)+(n-1)r\gamma+r\right)}$$

is determined by the sign of $\ln\left(\frac{pr(n\gamma-\gamma+1)}{(1-\beta)(1-p)(n\alpha-\alpha+1)}\right)$. This ln term is nothing but the left hand side of the Profitability condition.

Partner i's best response is the maximum of 0 and this expression.

Finally, solving the system in Expression 2 gives us

$$s_i^*=\frac{\left(n(r-\beta+1)\frac{1}{a_i}-(r-\alpha-\beta-r\gamma+\alpha\beta+1)\left(\sum_N\frac{1}{a_j}\right)\right)}{n(r-\beta+1)(\alpha+r\gamma-\alpha\beta)}\ln\left(\frac{pr(n\gamma-\gamma+1)}{(1-\beta)(1-p)(n\alpha-\alpha+1)}\right)\tag{3}$$

for each $i\in N$ (which is Expression 1 of Proposition 2). Under certain conditions, this expression will give us the unique Nash equilibrium of the partnership game.

Proof. (Proposition 1)

Case 1: The partnership agreement is $PE[\gamma,\alpha]$ such that $\max\{\alpha,\gamma\}>0$.

($\Longrightarrow$) Assume $PE[\gamma,\alpha]$ is acceptable for N. We want to show that both Profitability and Homogeneity conditions are satisfied. First suppose Profitability is violated. Then, for each $i\in N$ and for all s_{-i}, $\sigma_i(s_{-i})<0$. (This is because, as noted above, the constant term and the slope are both negative in Expression 2). Thus, the unique Nash equilibrium is $s=(0,...,0)$, contradicting acceptability of $PE[\gamma,\alpha]$. Next, suppose Profitability is satisfied but Homogeneity is violated. Then,

$$\left(n(r-\beta+1)\frac{1}{a_n}-(r-\alpha-\beta-r\gamma+\alpha\beta+1)\left(\sum_N\frac{1}{a_j}\right)\right)\leq 0$$

[11] This expression is equal to 0 if and only if $\alpha=\gamma=1$ and equal to 1 if and only if $\alpha=\gamma=0$. The former is trivial. To see the latter, note that

$$\frac{r(1-\gamma)+(1-\beta)(1-\alpha)}{(1-\alpha+n\alpha)(1-\beta)+(n-1)r\gamma+r}\leq 1$$

simplifies to

$$0\leq n\alpha(1-\beta)+nr\gamma,$$

the boundary achieved if and only if $\alpha=\gamma=0$.

and thus $s_n^* < 0$. This implies that agent n's Nash equilibrium contribution is zero, contradicting acceptability of $PE[\gamma, \alpha]$.

($\Longleftarrow$) Assume both Profitability and Homogeneity conditions are satisfied. We want to show that $PE[\gamma, \alpha]$ is acceptable for N. Remember that s^* in Expression 3 is the solution to the system of equations in Expression 2. By Profitability, we have

$$\ln\left(\frac{pr\,(n\gamma - \gamma + 1)}{(1-\beta)(1-p)(n\alpha - \alpha + 1)}\right) > 0$$

and by Homogeneity, we have

$$\left(n\,(r - \beta + 1)\,\frac{1}{a_n} - (r - \alpha - \beta - r\gamma + \alpha\beta + 1)\left(\sum_N \frac{1}{a_j}\right)\right) > 0.$$

This guarantees that $s^* > 0$. It is therefore the unique Nash equilibrium of the partnership game and thus, $PE[\gamma, \alpha]$ is acceptable.

Case 2: The partnership agreement is $PE[0, 0]$.

($\Longrightarrow$) Assume $PE[0, 0]$ is acceptable for N. We want to show that Profitability holds and Homogeneity holds with a weak inequality. First suppose Profitability is violated. Then, as noted above, the unique Nash equilibrium is $s = (0, ..., 0)$, contradicting acceptability of $PE[0, 0]$. Next, suppose Profitability is satisfied but Homogeneity is violated. Since $\alpha = \gamma = 0$, the condition becomes

$$\frac{1}{a_n} < \frac{1}{n}\left(\sum_N \frac{1}{a_j}\right).$$

This implies, $a_1 < a_n$. Again due to $\alpha = \gamma = 0$, Expression 2 simplifies to

$$s_i = \sigma_i(s_{-i}) = \frac{n \ln\left(\frac{pr}{(1-\beta)(1-p)}\right)}{a_i(1 - \beta + r)} - \left(\sum_{N \setminus i} s_j\right). \tag{4}$$

Since $a_1 < a_n$ then, agent n picks zero contributions in equilibrium, contradicting acceptability of $PE[0, 0]$.

($\Longleftarrow$) Assume both the Profitability and the weaker form of Homogeneity are satisfied. We want to show that $PE[0, 0]$ is acceptable for N. By the weaker form of Homogeneity,

$$\frac{1}{a_n} \geq \frac{1}{n}\left(\sum_N \frac{1}{a_j}\right)$$

which in turn implies $a_1 = ... = a_n$. Thus, the best response expression of every agent i can be written as

$$s_i = \sigma_i\left(s_{-i}\right) = \frac{n\ln\left(\frac{pr}{(1-\beta)(1-p)}\right)}{a_n\left(1-\beta+r\right)} - \left(\sum_{N\backslash i} s_j\right).$$

By Profitability, we have $\frac{n\ln\left(\frac{pr}{(1-\beta)(1-p)}\right)}{a_n(1-\beta+r)} > 0$. Thus, all $s^* \geq 0$ such that $\sum_N s_i^* = \frac{n\ln\left(\frac{pr}{(1-\beta)(1-p)}\right)}{a_n(1-\beta+r)}$ is a Nash equilibrium. Since a continuum of these equilibria satisfy $s^* > 0$, we conclude that $PE\left[0,0\right]$ is acceptable. ■

Proof. (Proposition 2)

Case 1: The partnership agreement is $PE\left[\gamma,\alpha\right]$ such that $\max\{\alpha,\gamma\} > 0$.

Assume that $PE[\gamma,\alpha]$ is acceptable for N. By Proposition 1 then, both Profitability and Homogeneity conditions are satisfied. We want to show that the resulting partnership game has a unique Nash equilibrium s^* which is given by Expression 1 (also shown in Expression 3). Note that this expression is the solution to the system in Expression 2. By Profitability, we have $\ln\left(\frac{pr(n\gamma-\gamma+1)}{(1-\beta)(1-p)(n\alpha-\alpha+1)}\right) > 0$ and by Homogeneity, we have

$$\left(n\left(r-\beta+1\right)\frac{1}{a_i} - \left(r-\alpha-\beta-r\gamma+\alpha\beta+1\right)\left(\sum_N \frac{1}{a_j}\right)\right) > 0.$$

This guarantees that that $s^* > 0$. It is thus the unique Nash equilibrium of the partnership game under $PE\left[\gamma,\alpha\right]$.

Case 2: The partnership agreement is $PE\left[0,0\right]$.

Assume $PE\left[0,0\right]$ is acceptable for N. By Proposition 1 then, both the Profitability and the weaker form of Homogeneity are satisfied. The ($\Longleftarrow$) part in Case 2 of the previous proof then shows that all $s^* \geq 0$ such that $\sum_N s_i^* = \frac{n\ln\left(\frac{pr}{(1-\beta)(1-p)}\right)}{a_n(1-\beta+r)}$ is a Nash equilibrium of the partnership game. ■

Proof. (Corollary 3) In the Expression 2, the slope is:

$$-\frac{r\left(1-\gamma\right)+\left(1-\beta\right)\left(1-\alpha\right)}{\left(1-\alpha+n\alpha\right)\left(1-\beta\right)+\left(n-1\right)r\gamma+r}.$$

If this expression is zero, the best response of partner i is independent of s_{-i}, making it a strictly dominant strategy. Now note that the denominator of this expression is always positive. And its numerator $r\left(1-\gamma\right)+\left(1-\beta\right)\left(1-\alpha\right) = 0$ if and only if $\alpha = \gamma = 1$. Therefore, $PE\left[1,1\right]$ is the only partnership agreement that always induces a dominant strategy equilibrium. ■

Proof. (Theorem 1) Total contribution is

$$\sum s_i^* = \frac{1}{r-\beta+1}\left(\sum \frac{1}{a_i}\right)\ln\left(\frac{pr\,(n\gamma-\gamma+1)}{(1-\beta)\,(1-p)\,(n\alpha-\alpha+1)}\right).$$[12]

The derivative of this expression with respect to α is

$$\begin{aligned}\frac{\partial\left(\frac{1}{(r-\beta+1)}\left(\sum_N \frac{1}{a_j}\right)\ln\left(\frac{pr(n\gamma-\gamma+1)}{(1-\beta)(1-p)(n\alpha-\alpha+1)}\right)\right)}{\partial\alpha} &= \\ -\left(\sum_N \frac{1}{a_j}\right)\frac{n-1}{(n\alpha-\alpha+1)\,(r-\beta+1)} &< 0.\end{aligned}$$

Thus, a decrease in the fraction of negative surplus allocated proportionally increases total contributions. Next, let us look at the derivative of this expression with respect to n:

$$\begin{aligned}&\frac{\partial\left(-\left(\sum_N \frac{1}{a_j}\right)\frac{n-1}{(n\alpha-\alpha+1)(r-\beta+1)}\right)}{\partial n} \\ = \; & -\frac{\left(\sum_N \frac{1}{a_j}\right)}{(n\alpha-\alpha+1)^2\,(r-\beta+1)} - \frac{n-1}{(n\alpha-\alpha+1)\,(r-\beta+1)}\frac{\partial\left(\sum_N \frac{1}{a_j}\right)}{\partial n} < 0.\end{aligned}$$

That is, the above derivative is increasing in absolute value as the number of agents increases. This implies that, in larger partnerships, switching from proportionality to equal shares of negative surplus has a greater effect.

Now, let us look at the derivative of total contributions respect to γ :

$$\begin{aligned}\frac{\partial\left(\frac{1}{(r-\beta+1)}\left(\sum_N \frac{1}{a_j}\right)\ln\left(\frac{pr(n\gamma-\gamma+1)}{(1-\beta)(1-p)(n\alpha-\alpha+1)}\right)\right)}{\partial\gamma} &= \\ \left(\sum_N \frac{1}{a_j}\right)\frac{n-1}{(n\gamma-\gamma+1)\,(r-\beta+1)} &> 0.\end{aligned}$$

So, as the positive-surplus rule gets closer to proportionality, equilibrium total contribution increases. Next, let us check how this derivative changes with n:

$$\begin{aligned}&\frac{\partial\left(\left(\sum_N \frac{1}{a_j}\right)\frac{n-1}{(n\gamma-\gamma+1)(r-\beta+1)}\right)}{\partial n} \\ = \; & \frac{\left(\sum_N \frac{1}{a_j}\right)}{(n\gamma-\gamma+1)^2\,(r-\beta+1)} + \frac{n-1}{(n\gamma-\gamma+1)\,(r-\beta+1)}\frac{\partial\left(\sum_N \frac{1}{a_j}\right)}{\partial n} > 0.\end{aligned}$$

[12] Note that, this expression gives total contribution when $\alpha = \gamma = 0$ as well. Even though there is multiplicity of equilibria in this case, they all have the same total contribution level given by this expression.

That is, the above derivative is increasing in absolute value as the number of agents increases. This implies that, in larger partnerships, switching from equal shares to proportional allocation of positive surplus has a greater effect. ■

Proof. (Proposition 4) Introducing the equilibrium contribution into partner i's utility function, we obtain

$$U_i^{PE[\gamma,\alpha]}(s^*) = \begin{pmatrix} -p\left(\frac{pr(n\gamma-\gamma+1)}{(1-\beta)(1-p)(n\alpha-\alpha+1)}\right)^{\frac{-r\gamma}{\alpha+r\gamma-\alpha\beta}} \\ -(1-p)\left(\frac{pr(n\gamma-\gamma+1)}{(1-\beta)(1-p)(n\alpha-\alpha+1)}\right)^{\frac{\alpha(1-\beta)}{\alpha+r\gamma-\alpha\beta}} \end{pmatrix} \times \left(\frac{pr\left(n\gamma-\gamma+1\right)}{\left(1-\beta\right)\left(1-p\right)\left(n\alpha-\alpha+1\right)}\right)^{\frac{ra_i(1-\beta)(\gamma-\alpha)}{n(r-\beta+1)(\alpha+r\gamma-\alpha\beta)}\left(\sum\frac{1}{a_j}\right)}$$

Looking at the power of the second term, $\frac{ra_i(1-\beta)(\gamma-\alpha)}{n(r-\beta+1)(\alpha+r\gamma-\alpha\beta)}\left(\sum\frac{1}{a_j}\right)$, we note that all its components have determinate signs except $\gamma-\alpha$ whose sign determines the effect of a_i on $U_i^{PE[\gamma,\alpha]}(s^*)$. If $\gamma-\alpha>0$, an increase in a_i decreases $U_i^{PE[\gamma,\alpha]}(s^*)$. As a result, Partner 1 receives the highest utility and Partner n, the lowest. The welfare ordering of the partners is exactly the opposite when $\gamma-\alpha<0$. And if $\gamma-\alpha=0$, a_i does not affect $U_i^{PE[\gamma,\alpha]}(s^*)$. Thus, all agents receive the same utility. ■

Proof. (Theorem 2) Proposition 4 establishes that

$$\mathcal{EG}^{PE[\gamma,\alpha]}(p,r,\beta,a_1,...,a_n) = \begin{cases} U_1^{PE[\gamma,\alpha]}(s^*) & if \quad \gamma<\alpha \\ U_n^{PE[\gamma,\alpha]}(s^*) & if \quad \gamma>\alpha \\ U_1^{PE[\gamma,\alpha]}(s^*)=...=U_n^{PE[\gamma,\alpha]}(s^*) & if \quad \gamma=\alpha \end{cases} \min_{i\in N} U_i(\epsilon\left(G^{PE[\gamma,\alpha]}\right)).$$

We will treat each case separately. First, assume $\gamma=\alpha$. In this case, the individual utility functions simplify to

$$U_i^{PE[\gamma,\alpha]}(s^*) = \left(-p\left(\frac{pr}{(1-\beta)(1-p)}\right)^{\frac{-r}{1+r-\beta}} - (1-p)\left(\frac{pr}{(1-\beta)(1-p)}\right)^{\frac{(1-\beta)}{1+r-\beta}}\right)$$

Two observations are in line. First, Partner i's equilibrium utility is independent of a_i. Therefore, all partners receive identical utility. Second, the expression is independent of the common value of $\gamma=\alpha$. That is, all $\gamma=\alpha$ partnership agreements produce the same level of egalitarian social welfare. This establishes the second sentence of the theorem.

To see the first sentence, next assume $\gamma \neq \alpha$. Letting

$$
\begin{aligned}
\left(\frac{r(1-\beta)}{n(r-\beta+1)}\left(\sum_j \frac{a_i}{a_j}\right)\right) &= A \text{ and} \\
\left(\frac{pr(n\gamma-\gamma+1)}{(1-\beta)(1-p)}\right) &= B,
\end{aligned}
$$

the derivative of partner i's utility with respect to α can be written as

$$
\begin{aligned}
&\frac{\partial U_i^{PE[\gamma,\alpha]}(s^*)}{\partial \alpha} \\
= \quad & \frac{p\gamma(n\alpha-\alpha+1)^2((r-\beta+1)A-r(1-\beta))\ln\left(\frac{B}{n\alpha-\alpha+1}\right)}{(n\alpha-\alpha+1)^2(\alpha+r\gamma-\alpha\beta)^2\left(\frac{B}{n\alpha-\alpha+1}\right)^{\frac{(r\gamma-(\gamma-\alpha)A)}{(\alpha+r\gamma-\alpha\beta)}}} \\
& -(n-1)(\alpha+r\gamma-\alpha\beta) \\
& \times\frac{((A\alpha+(r-A)\gamma)p\alpha n+((A-r)\gamma\alpha+(A+B(1-\beta))\alpha+\gamma(r-A)-A\alpha^2)p-B\alpha(1-\beta))}{(n\alpha-\alpha+1)^2(\alpha+r\gamma-\alpha\beta)^2\left(\frac{B}{n\alpha-\alpha+1}\right)^{\frac{(r\gamma-(\gamma-\alpha)A)}{(\alpha+r\gamma-\alpha\beta)}}} \\
& +\frac{\left(A\gamma(1-p)(n\alpha-\alpha+1)^2(r-\beta+1)-r\gamma(1-\beta)(1-p)(n\alpha-\alpha+1)^2\right)\ln\left(\frac{B}{n\alpha-\alpha+1}\right)}{(n\alpha-\alpha+1)^2(\alpha+r\gamma-\alpha\beta)^2\left(\frac{B}{n\alpha-\alpha+1}\right)^{-\frac{(\alpha(1-\beta)+(\gamma-\alpha)A)}{(\alpha+r\gamma-\alpha\beta)}}} \\
& +\frac{A(\gamma-\alpha)(n-1)(1-p)(n\alpha-\alpha+1)(\alpha+r\gamma-\alpha\beta)}{(n\alpha-\alpha+1)^2(\alpha+r\gamma-\alpha\beta)^2\left(\frac{B}{n\alpha-\alpha+1}\right)^{-\frac{(\alpha(1-\beta)+(\gamma-\alpha)A)}{(\alpha+r\gamma-\alpha\beta)}}}
\end{aligned}
$$

After eliminating terms that do not change the sign of this expression we obtain the following simpler expression:

$$
\begin{aligned}
&\gamma(n\alpha-\alpha+1)((n-1)p\alpha+1)((r-\beta+1)A-r(1-\beta))\ln\left(\frac{B}{n\alpha-\alpha+1}\right) \\
&-(n-1)(\alpha+r\gamma-\alpha\beta)((\alpha-\gamma)A((1-p)B+(1-\alpha+n\alpha)p)+pr\gamma(1-\alpha+n\alpha)-B\alpha(1-p)(1
\end{aligned}
$$

After inserting in the expressions for A and B, we obtain

$$
\begin{aligned}
&\gamma r(1-\beta)(r-\beta+1)(n\alpha-\alpha+1)((n-1)p\alpha+1)\left(\left(\sum_j \frac{a_i}{a_j}\right)-n\right) \\
&\times\ln\left(\frac{pr(n\gamma-\gamma+1)}{(1-\beta)(1-p)(n\alpha-\alpha+1)}\right) \\
&-(n-1)(\alpha+r\gamma-\alpha\beta) \\
&\times\left(pr(\alpha-\gamma)\left(\sum_j \frac{a_i}{a_j}\right)(r(n\gamma-\gamma+1)+(1-\beta)(1-\alpha+n\alpha))-n(r-\beta+1)pr(\alpha-\gamma)\right).
\end{aligned}
$$

The sign of this expression will determine the effect of α on U_i. Since we are interested in egalitarian social welfare, let us consider the two cases separately. First assume $\gamma > \alpha$ and $i = n$. Then the above expression becomes

$$\begin{aligned}
&\gamma r\left(1-\beta\right)\left(r-\beta+1\right)\left(n\alpha-\alpha+1\right)\left(\left(n-1\right)p\alpha+1\right)\left(\left(\sum_j \frac{a_n}{a_j}\right)-n\right)\\
&\times \ln\left(\frac{pr\left(n\gamma-\gamma+1\right)}{\left(1-\beta\right)\left(1-p\right)\left(n\alpha-\alpha+1\right)}\right)\\
&-\left(n-1\right)\left(\alpha+r\gamma-\alpha\beta\right)\\
&\times\left(pr\left(\alpha-\gamma\right)\left(\sum_j \frac{a_n}{a_j}\right)\left(r\left(n\gamma-\gamma+1\right)+\left(1-\beta\right)\left(1-\alpha+n\alpha\right)\right)-n\left(r-\beta+1\right)pr\left(\alpha-\gamma\right)\right)
\end{aligned}$$

Since $\left(\left(\sum_j \frac{a_n}{a_j}\right)-n\right) \geq 0$, the first term is nonnegative. And it is strictly positive unless $a_1 = ... = a_n$. The second term is also positive since

$$\left(pr\left(\alpha-\gamma\right)\left(\sum_j \frac{a_n}{a_j}\right)\left(r\left(n\gamma-\gamma+1\right)+\left(1-\beta\right)\left(1-\alpha+n\alpha\right)\right)-n\left(r-\beta+1\right)pr\left(\alpha-\gamma\right)\right) < 0.\text{[13]}$$

This establishes that egalitarian social welfare is increasing in α when $\gamma > \alpha$.

Next assume $\gamma < \alpha$ and $i = 1$. Then, the above expression becomes

$$\begin{aligned}
&\gamma r\left(1-\beta\right)\left(r-\beta+1\right)\left(n\alpha-\alpha+1\right)\left(\left(n-1\right)p\alpha+1\right)\left(\left(\sum_j \frac{a_1}{a_j}\right)-n\right)\\
&\times \ln\left(\frac{pr\left(n\gamma-\gamma+1\right)}{\left(1-\beta\right)\left(1-p\right)\left(n\alpha-\alpha+1\right)}\right)\\
&-\left(n-1\right)\left(\alpha+r\gamma-\alpha\beta\right)\\
&\times\left(pr\left(\alpha-\gamma\right)\left(\sum_j \frac{a_1}{a_j}\right)\left(r\left(n\gamma-\gamma+1\right)+\left(1-\beta\right)\left(1-\alpha+n\alpha\right)\right)-n\left(r-\beta+1\right)pr\left(\alpha-\gamma\right)\right)
\end{aligned}$$

The first term is nonpositive since $\left(\left(\sum_j \frac{a_1}{a_j}\right)-n\right) \leq 0$. And it is strictly negative unless $a_1 = ... = a_n$. The second term is also negative since

$$\left(pr\left(\alpha-\gamma\right)\left(\sum_j \frac{a_1}{a_j}\right)\left(r\left(n\gamma-\gamma+1\right)+\left(1-\beta\right)\left(1-\alpha+n\alpha\right)\right)-n\left(r-\beta+1\right)pr\left(\alpha-\gamma\right)\right) > 0.$$

[13] For brevity of presentation, calculations that prove this and similar secondary claims have been skipped. However, they all are available from the authors upon request.

This establishes that egalitarian social welfare is decreasing in α when $\gamma < \alpha$.

Similar calculations show that the egalitarian social welfare is decreasing in γ when $\gamma > \alpha$ and increasing in γ otherwise. These observations, together, prove the first statement of the theorem. ■

Claim 1 *All $PE\,[x, x]$ agreements induce the same amount of total contributions. However, a partner more (less) risk averse than average responds to an increase in x by increasing (decreasing) his contributions.*

Proof. Under $PE\,[x, x]$, the total contribution expression (used in the proof of Theorem 1) simplifies to

$$\sum s_i^* = \frac{1}{r - \beta + 1} \left(\sum \frac{1}{a_i} \right) \ln \left(\frac{pr}{(1-\beta)(1-p)} \right).$$

Note that the expression is independent of x, proving the first claim.

For the second claim, note that Expression 1 simplifies to

$$s_i^* = \frac{\left(n \frac{1}{a_i} - (1-x) \left(\sum_N \frac{1}{a_j} \right) \right) \ln \left(\frac{pr}{(1-p)(1-\beta)} \right)}{nx\,(r + 1 - \beta)}$$

under $PE\,[x, x]$. Taking the derivative of this expression with respect to x, we obtain

$$\left(\frac{1}{n} \left(\sum_N \frac{1}{a_j} \right) - \frac{1}{a_i} \right) \frac{\ln \left(\frac{pr}{(1-p)(1-\beta)} \right)}{x^2\,(r - \beta + 1)}.$$

The second part of the expression is positive (by Profitability). Thus, the sign is determined by the first part. If agent i is more (less) risk averse than average, this term is positive (negative), the desired conclusion. ■